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Using MLM, NWS and LLS to Estimate of a Multivariate Regression Functions Based on the Skewed Heavy Tail Distribution Family

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Abstract

The families of probability distributions with heavy tails are considered one of the most essential continuous distributions that have broad uses in various areas of life, especially in areas related to economics, which is concerned with the subject of oil prices and securities, so the research was estimated two types of regression functions, represented by the multivariable parametric, non-parametric regression function, depending on the Matrix-Variate Variance Gamma(M-VVG) distribution and Matrix-Variate Normal Inverse Gaussian(M-VNIG) distribution for the error of models. As the multivariate non-parametric regression model was converted into a linear model based on the local polynomial smoother and through the classical method, multivariate Nadarya Watson smoother (NW-S) and the multivariate local linear smoother (LL-S) were obtained, as well as estimating the multivariate parametric regression function through the use of the maximum likelihood method (ML-M). The results were applied to actual data represented by Brent crude oil price data for the period from (2/11/2020) to (8/12/2020) measured in US dollars, and through the results of the Matlab programming and depending on the MSE standard, we note the superiority of the multivariate (NW-S) for the multivariate non-parametric regression function and for the error that follows a (M-VVG) distribution and for a Gauss kernel function, the value of the criterion was (MSE=0.2314), followed by the (M-VNIG) where the criterion (MSE=0.5601). The superiority of the identical error distributions for the multivariate parametric regression function, as well as the value of the criterion, was (MSE=0.4321,0.6433) respectively.

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Introduction

Brent crude is the primary and commercial classification of light sweet crude oil, which is used as a criterion for oil purchase prices globally due to its low density of sulfur. It was called light, and this type of oil is extracted from the North Sea and is also known as the Brent mixture or the London mix. It consists of about 15 oil varieties, most produced from Scotland and Norway. Its value is used to price two-thirds of the world's traded crude oil imports.

Multivariate regression models are one of the most important models used in analyzing such economic data, as they interest statisticians and scientists due to their flexibility. The use of parametric models and subsequent estimation methods requires many initial conditions that these models must meet to represent the population under study,

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which prompted both [5] the search for more flexible models than parametric models represented by non-parametric models. The non-parametric regression function was estimated using the (NW-S) in two cases (the first case using a fixed bandwidth parameter and the second variable) through simulation with different sample models and sizes [5], and through the experimental results, the (NW-S) excels at the state of the fixed bandwidth parameter for the first and second model, and at the case of the variable bandwidth parameter, the (NW-S) showed high efficiency of the third model. But in fact, there are cases in which the error of the models has a probability distribution with a tail or heavy ends; in such a case, it is more appropriate to pay attention to alternative distributions from the Matrix-Variate Normal (M-VN) distribution, i.e., the mixed distribution, and from these distributions is a Matrix-Variate Variance Gamma (M-VVG), and a Matrix- Variate Normal Inverse Gaussian (M-VNIG) distributions [1,16,17,19].

He studied [9] the essential properties of the symmetric heavy tail distribution with its particular states. He confirmed that mixed distributions such as the mixed multivariate normal distribution and the multivariate T distribution are exceptional cases of it and that it is a distribution resulting from mixing the normal multivariate probability distribution and the Extension Inverse Normal (EIN) probability distribution, as well as its applications in the Bayesian analyses of the parametric regression model with normal error and under the assumption that the previous distribution of the variance parameter is the (EIN) probability distribution. In (2004), [10] skewed heavy tail distribution was generalized to the symmetric heavy tail distribution and the study of some of its essential properties and exceptional cases, and that it is a distribution resulting from mixing the normal twisted matrix distribution and the matrix (EIN) distribution, as well as its applications in the Bayesian analyses of the multivariate parametric regression model with normal error assuming the matrix (EIN) distribution as a previous distribution of the variance matrix parameter.

In [21], the authors demonstrated the truncated non-parametric regression spline model for longitudinal data when the random error matrix follows a (M-VN) distribution, as well as estimating the non-parametric function based on the (NW-S), (LL-S). The estimation of the scale matrix Σ based on the (ML-M) and concluded that the (NW-S) outperforms (LL-S) in smoothing the non-parametric regression function based on several comparison criteria.

The second section dealt with the paper's aim. The third section dealt with the distribution of the (M-VVG) and (M-VNIG). In the fourth section, the model of Non-normal Multivariate Parametric Regression (Non-NMPR) was described. In the fifth section, the multivariate non-parametric regression model has been converted into a linear multivariate model based on the concept of the Taylor series. The sixth section included a description of the model of Non-normal Multivariate Non-parametric Regression (Non-NMNPR). In the seventh section, some kernel functions and the thumb rule were mentioned to choose the smoother parameter. In the eighth section, classical methods of non-normal regression (Non-NR) functions were used in multiple ways, and the ninth section included the application of what was reached on actual data. In the last section, the most prominent conclusions of the study were presented.

2. Paper aim.

The paper aims to find the estimators of multivariate parametric and non-parametric regression functions when the error term of the models follows Skewed heavy-tailed probability distributions represented by the (M-VVG, M-VNIG) distributions, in addition to the application to actual data related to Brent crude oil.

3. Skewed Heavy-Tails Distributions.

- **(M-VVG)** distribution is one of the most essential skewed heavy-tailed mixed continuous probability distributions. It results from mixing the M-VN and Gamma with parameters (ϕ,ϕ)—the probability density function defined by reference [5].

$$f(\mathrm{Y})=\frac{2\left(\frac{1}{2}\right)^{\frac{\mathrm{nR}}{2}}\left(\int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt\right)^{-\mathrm{nR}}(\Gamma(\phi))^{-1}\left(\mathrm{trace}(\mathrm{Y}-\mu)^{\mathrm{T}}(\mathrm{Y}-\mu)\,\Sigma^{-1}\right)^{\frac{\phi}{2}-\frac{\mathrm{nR}}{4}}}{\phi^{-\phi}\;(\det(\Sigma))^{\frac{\mathrm{n}}{2}}\;\;\mathrm{e}^{-\mathrm{trace}\,(\mathrm{Y}-\mu)^{\mathrm{T}}\,\omega\,\Sigma^{-1}}\left(\mathrm{trace}\;\omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+2\phi\right)^{\frac{\phi}{2}-\frac{\mathrm{nR}}{4}}}$$

$$* \mathrm{H}_{\phi-\frac{nR}{2}}\left(\mathrm{sqrt}\left(\left(\mathrm{trace}\ (\mathrm{Y}-\mu)^{\mathrm{T}}(\mathrm{Y}-\mu)\ \Sigma^{-1}\right)\left(\mathrm{trace}\ \omega^{\mathrm{T}}\ \omega\ \Sigma^{-1}+2\phi\right)\right)\right) \qquad (1)$$

Whereas:

$\phi > 0$

$\mathrm{H}_{\gamma}(\mathrm{sqrt}(\alpha\,\vartheta))$: Hankel function of order γ, defined by: [3][4]

$$\mathrm{H}_{\gamma}(\mathrm{sqrt}(\alpha\,\vartheta)) = \int_0^{\infty} \frac{\mathrm{x}^{-(1-\gamma)}}{2\, e^{\frac{-1}{2}(\mathrm{sqrt}(\alpha\,\vartheta))\left[-\frac{1}{\mathrm{x}}-\mathrm{x}\right]}}\mathrm{dx} \qquad (2)$$

ω: Skewed matrix with a dimension n×R.

μ: location matrix with n×R.

Σ: measurement matrix with dimension R×R.

- **(M-VNIV)** distribution is one of the most essential skewed heavy-tailed mixed continuous probability distributions and is the result of mixing the M-VN and Inverse Gamma with parameters $(1,\tilde{\phi})$. The probability density function is defined by reference [1].

$$f(\mathrm{Y}) = \frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}+1}\left(\int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt\right)^{-nR+2}\left(\mathrm{trace}(\mathrm{Y}-\mu)^{\mathrm{T}}(\mathrm{Y}-\mu)\ \Sigma^{-1}+1\right)^{-\left(\frac{1+nR}{4}\right)}}{(\det(\Sigma))^{\frac{n}{2}}\ \ e^{-\mathrm{trace}\ (\mathrm{Y}-\mu)^{\mathrm{T}}\ \omega\ \Sigma^{-1}-\tilde{\phi}}\left(\mathrm{trace}\ \omega^{\mathrm{T}}\ \omega\ \Sigma^{-1}+\tilde{\phi}^2\right)^{-\left(\frac{1+nR}{4}\right)}}$$

$$* \mathrm{H}_{-\left(\frac{1+nR}{2}\right)}\left(\mathrm{sqrt}\left(\left(\mathrm{trace}\ (\mathrm{Y}-\mu)^{\mathrm{T}}(\mathrm{Y}-\mu)\ \Sigma^{-1}+1\right)\left(\mathrm{trace}\ \omega^{\mathrm{T}}\ \omega\ \Sigma^{-1}+\tilde{\phi}^2\right)\right)\right) \qquad (3)$$

Whereas:

$\tilde{\phi} > 0$

4. Description of Non-NMPR Model.

(EHMPR) model is described according to the following equation [4]:

$$\mathrm{Y}_{ik} = \mathrm{X}_i'\theta_k + \mathrm{E}_{ik} \quad i = 1,2,\dots,n \ \ , k = 1,2,\dots,R \qquad (4)$$

Equation (4) can be rewritten in matrix form as follows: [10]

$$\mathrm{Y}_{n\times R} = \mathrm{X}_{n\times p+1}\theta_{p+1\times R} + \mathrm{E}_{n\times R} \qquad (5)$$

Assuming that the random error matrix (E) follows a skewed heavy-tails (M-VVG) distribution and (M-VNIG) where the probability density functions can be found using the concept of Bayes' theorem from the mixed (M-VN) distribution and the one-variable gamma (ϕ,ϕ) distribution and inverse gamma $(1,\tilde{\phi})$ distribution respectively, as follows:[1,10,14]

The probability density function of the random error matrix conditioned by the random variable (τ) takes the following form:[6]

$$f(\mathrm{E}|\tau) = (2\tau)^{\frac{-nR}{2}}\,\Gamma\left(\frac{1}{2}\right)^{-nR}\ \det(\Sigma)^{\frac{-n}{2}}\ e^{-\frac{1}{2\tau}\mathrm{trace}\ (\mathrm{E}-\omega\tau)^{\mathrm{T}}(\mathrm{E}-\omega\tau)\Sigma^{-1}} \qquad (6)$$

Equation (6) represents the distribution of a mixed normal matrix distribution, and the probability density function of the random variable (τ) is defined by reference [5]:

$$if\ \tau\sim \mathrm{g}(\phi,\phi)\ then\ P(\tau) = \frac{\phi^{\phi}\ \tau^{\phi-1}}{\Gamma(\phi)\ e^{(\phi\tau)}}\ ,\tau > 0$$

$$if\ \tau\sim \mathrm{ig}(1,\tilde{\phi})\ then\ P(\tau) = \frac{\tilde{\phi}\ \tau^{-2}}{e^{\left(\frac{\tilde{\phi}}{\tau}\right)}}\ ,\tau > 0 \qquad (7)$$

According to the concept of Bayes' theorem, the probability distribution of the unconditional random error matrix with (τ) is as follows:

$$f(\mathrm{E})=\int_{0}^{\infty} f(\mathrm{E}|\tau)\, P(\tau)\ \ \mathrm{d}\tau$$

$$f(\mathrm{E})=\frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}}\left(\int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt\right)^{-\,nR}(\Gamma(\phi))^{-1}\left(\text{ trace } \mathrm{E}^{\mathrm{T}}\mathrm{E}\,\Sigma^{-1}\right)^{\frac{\phi}{2}-\frac{nR}{4}}}{\phi^{-\phi}\ (\det(\Sigma))^{\frac{n}{2}}\ \ e^{-\text{ trace } \mathrm{E}^{\mathrm{T}}\,\omega\,\Sigma^{-1}}\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+2\phi\right)^{\frac{\phi}{2}-\frac{nR}{4}}}$$

$$*\ \mathrm{H}_{\phi-\frac{nR}{2}}\left(\operatorname{sqrt}\left(\left(\text{ trace } \mathrm{E}^{\mathrm{T}}\mathrm{E}\,\Sigma^{-1}\right)\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+2\phi\right)\right)\right) \qquad (8)$$

Equation (8) represents the distribution of the skewed heavy-tails (M-VVG) of (E) [1, 17]:

Return to the model in equation (4) and assuming the random error term follows an (M-VNIG) distribution, and with the same steps to find equation (8), then:

$$f(\mathrm{Y})=\frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}+1}\left(\int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt\right)^{-\,nR+2}\left(\text{ trace}\mathrm{E}^{\mathrm{T}}\mathrm{E}\,\Sigma^{-1}+1\right)^{-\left(\frac{1+nR}{4}\right)}}{(\det(\Sigma))^{\frac{n}{2}}\ \ e^{-\text{ trace } \mathrm{E}^{\mathrm{T}}\,\omega\,\Sigma^{-1}-\tilde{\phi}}\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+\tilde{\phi}^{2}\right)^{-\left(\frac{1+nR}{4}\right)}}$$

$$*\ \mathrm{H}_{-\left(\frac{1+nR}{2}\right)}\left(\operatorname{sqrt}\left(\left(\text{ trace } \mathrm{E}^{\mathrm{T}}\mathrm{E}\,\Sigma^{-1}+1\right)\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+\tilde{\phi}^{2}\right)\right)\right) \qquad (9)$$

Equation (9) represents the distribution of the skewed heavy-tails (M-VNIV) of (E), Since the matrix of observations of the response variable of (Y) in equation (4) is a linear combination in terms of the random error matrix of the model that follows the distribution of skewed heavy-tails (M-VVG) and (M-VNIG), the probability distribution of the response variables matrix follows the skewed heavy- tails (M-VVG) and (M-VNIG) respectively as follows:

$$f(\mathrm{Y}|\tau)=\left(\frac{1}{\tau}\right)^{\frac{nR}{2}}(2)^{\frac{-\,nR}{2}}\,\Gamma\left(\frac{1}{2}\right)^{-nR}|\Sigma|^{\frac{-n}{2}}\ \ e^{-\frac{1}{2\,\tau}\text{trace }(\mathrm{Y}-\mathrm{X}\theta-\omega\tau)^{\mathrm{T}}(\mathrm{Y}-\mathrm{X}\theta-\omega\tau)\Sigma^{-1}} \qquad (10)$$

Based on the concept of Bayes' theorem, the probability density function of (Y) unconditional to the random variable (τ) is as follows:

$$f(\mathrm{Y})=\frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}}\left(\int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt\right)^{-\,nR}(\Gamma(\phi))^{-1}\left(\text{ trace}(\mathrm{Y}-\mathrm{X}\theta)^{\mathrm{T}}(\mathrm{Y}-\mathrm{X}\theta)\,\Sigma^{-1}\right)^{\frac{\phi}{2}-\frac{nR}{4}}}{\phi^{-\phi}\ (\det(\Sigma))^{\frac{n}{2}}\ \ e^{-\text{ trace }(\mathrm{Y}-\mathrm{X}\theta)^{\mathrm{T}}\,\omega\,\Sigma^{-1}}\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+2\phi\right)^{\frac{\phi}{2}-\frac{nR}{4}}}$$

$$*\ \mathrm{H}_{\phi-\frac{nR}{2}}\left(\operatorname{sqrt}\left(\left(\text{ trace }(\mathrm{Y}-\mathrm{X}\theta)^{\mathrm{T}}(\mathrm{Y}-\mathrm{X}\theta)\,\Sigma^{-1}\right)\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+2\phi\right)\right)\right) \qquad (11)$$

Equation (11) represents the distribution of the skewed heavy-tails (M-VVG) of (Y).

$$f(\mathrm{Y})=\frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}+1}\left(\int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt\right)^{-\,nR+2}\left(\text{ trace}(\mathrm{Y}-\mathrm{X}\theta)^{\mathrm{T}}(\mathrm{Y}-\mathrm{X}\theta)\,\Sigma^{-1}+1\right)^{-\left(\frac{1+nR}{4}\right)}}{(\det(\Sigma))^{\frac{n}{2}}\ \ e^{-\text{ trace }(\mathrm{Y}-\mathrm{X}\theta)^{\mathrm{T}}\,\omega\,\Sigma^{-1}-\tilde{\phi}}\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+\tilde{\phi}^{2}\right)^{-\left(\frac{1+nR}{4}\right)}}$$

$$*\ \mathrm{H}_{-\left(\frac{1+nR}{2}\right)}\left(\operatorname{sqrt}\left(\left(\text{ trace }(\mathrm{Y}-\mathrm{X}\theta)^{\mathrm{T}}(\mathrm{Y}-\mathrm{X}\theta)\,\Sigma^{-1}+1\right)\left(\text{ trace } \omega^{\mathrm{T}}\,\omega\,\Sigma^{-1}+\tilde{\phi}^{2}\right)\right)\right) \qquad (12)$$

Equation (12) represents the distribution of the skewed heavy-tails (M-VNIG) of (Y).

5. Local Polynomial Smoother

This smoother is one of the good smoothers used in the non-parametric bandwidth and is preferred to the kernel smoother as it is used in static or random design. The following formula can represent the multivariate non-parametric regression model [11] [12]:

$$Y_{ik} = m_k(T_i) + E_{ik} \tag{13}$$

$$\begin{bmatrix} y_{11} & \cdots & y_{1R} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nR} \end{bmatrix} = \begin{bmatrix} m_1(T_1) & \cdots & m_R(T_1) \\ \vdots & \ddots & \vdots \\ m_1(T_n) & \cdots & m_R(T_n) \end{bmatrix} + \begin{bmatrix} E_{11} & \cdots & E_{1R} \\ \vdots & \ddots & \vdots \\ E_{n1} & \cdots & E_{nR} \end{bmatrix} \tag{14}$$

As $m_k(T_i)$ represents an unknown introductory function matrix with (R) of response variables and (T) of non-parametric explanatory variables and usually a continuous variable; from equation (13), we note that the multivariate non-parametric regression function is non-linear and based on the concept of Taylor series (Taylor expansion) has been converted into a linear multivariate function as follows[6]:

Suppose that the function (m) has a derivative of the rank (u) at point (t), and the points lie in the vicinity of point t. The concept of Taylor's expansion of the function (m) is as follows:

$$m(T_i) = \beta_0 + \beta_1(t - T_i) + \beta_2(t - T_i)^2 + \cdots + \beta_u(t - T_i)^u \tag{15}$$

$$m(T_i) = \sum_{l=0}^{u} \beta_l (t - T_i)^l \quad , \quad i = 1,2,\dots,n \tag{16}$$

Note that equation (16) above represents the regression function of an independent non-parametric variable and a dependent variable, and therefore, equation (16) for (R) has been generalized from the response variables as follows: [12]

$$m_k(T_i) = \sum_{l=0}^{u} \sum_{k=1}^{R} \beta_{lk} (t - T_i)^l \quad , \quad i = 1,2,\dots,n \tag{17}$$

We notice from equation (17) that the regression function has become linear, and therefore, the linear model can be written in matrix form:

$$Y = T\beta + E \tag{18}$$

Since:

$$T = \begin{bmatrix} 1 & (t - T_1) & (t - T_1)^2 & \dots & (t - T_1)^u \\ \vdots & & \vdots & & \vdots \\ 1 & (t - T_n) & (t - T_n)^2 & \dots & (t - T_n)^u \end{bmatrix}_{n\times u+1} ,$$

$$\beta = \begin{bmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0R} \\ & \vdots & \vdots & \vdots \\ \beta_{u1} & \beta_{u2} & \cdots & \beta_{uR} \end{bmatrix}_{u+1\times R}$$

6. Description of the Non-NMNPR Model:

We know that the multivariate non-parametric regression model is described according to equation (18), and assuming that the random error matrix of the model follows skewed heavy-tails (M-VVG) and (M-VNIG) distributions, and using the concept of Bayes' theory, the probability density function can be found from the mixed (M-VN) distribution and gamma (ϕ, ϕ) distribution and inverse gamma $(1, \tilde{\phi})$ distribution respectively, agencies [5][10]:

$$E|\tau = \sim M - VN_{n,R}(\omega\tau\, , \tau\,\Sigma\, , \varphi^{-1})$$

Whereas:

Φ: is a diagonal matrix representing the weights of the n×n kernel function.

$$\varphi = \mathrm{diag}\left(k_h(t - T_i)\right) \quad , \quad k_h(.) = \frac{1}{h}\, k\left(\frac{\cdot}{h}\right)$$

$k_h(.)$: Kernel functions and is defined as positive, continuous, symmetrical, and its integral is equal to the integer [2] [8].

h: The smoothing parameter is positive and more critical than the kernel function and can be selected according to the researcher's experience [2][7].

In the same way as the third topic, the probability distribution of the observations matrix (Y) is as follows:

$$f(\mathrm{Y}) = \frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}} \left(\int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt\right)^{-nR} (\Gamma(\phi))^{-1} \left(\mathrm{trace}(\mathrm{Y}-\mathrm{T}\beta)^{\mathrm{T}} \varphi\, (\mathrm{Y}-\mathrm{T}\beta)\, \Sigma^{-1}\right)^{\frac{\phi}{2}-\frac{nR}{4}}}{\phi^{-\phi}\, (\det(\Sigma))^{\frac{n}{2}} (\det(\varphi))^{-\frac{R}{2}}\; e^{-\mathrm{trace}\,(\mathrm{Y}-\mathrm{T}\beta)^{\mathrm{T}} \varphi\, \omega\, \Sigma^{-1}} \left(\mathrm{trace}\, \omega^{\mathrm{T}} \omega\, \Sigma^{-1} + 2\phi\right)^{\frac{\phi}{2}-\frac{nR}{4}}}$$

$$* \mathrm{H}_{\phi-\frac{nR}{2}}\left(\mathrm{sqrt}\left(\left(\mathrm{trace}\,(\mathrm{Y}-\mathrm{T}\beta)^{\mathrm{T}} \varphi\, (\mathrm{Y}-\mathrm{T}\beta)\, \Sigma^{-1}\right)\left(\mathrm{trace}\, \omega^{\mathrm{T}} \omega\, \Sigma^{-1} + 2\phi\right)\right)\right) \qquad (19)$$

The random (Y) in equation (19) follows a skewed heavy-tail distribution (M-VVG).

$$f(\mathrm{Y}) = \frac{2\left(\frac{1}{2}\right)^{\frac{nR}{2}+1} \left(\int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt\right)^{-nR+2} \left(\mathrm{trace}(\mathrm{Y}-\mathrm{T}\beta)^{\mathrm{T}} \varphi\, (\mathrm{Y}-\mathrm{T}\beta)\, \Sigma^{-1} + 1\right)^{-\left(\frac{1+nR}{4}\right)}}{(\det(\varphi))^{-\frac{R}{2}}\, (\det(\Sigma))^{\frac{n}{2}}\; e^{-\mathrm{trace}\,(\mathrm{Y}-\mathrm{T}\beta)^{\mathrm{T}} \varphi\, \omega\, \Sigma^{-1} - \tilde{\phi}} \left(\mathrm{trace}\, \omega^{\mathrm{T}} \omega\, \Sigma^{-1} + \tilde{\phi}^2\right)^{-\left(\frac{1+nR}{4}\right)}}$$

$$* \mathrm{H}_{-\left(\frac{1+nR}{2}\right)}\left(\mathrm{sqrt}\left(\left(\mathrm{trace}\,(\mathrm{Y}-\mathrm{T}\beta)^{\mathrm{T}} \varphi\, (\mathrm{Y}-\mathrm{T}\beta)\, \Sigma^{-1} + 1\right)\left(\mathrm{trace}\, \omega^{\mathrm{T}} \omega\, \Sigma^{-1} + \tilde{\phi}^2\right)\right)\right) \quad (20)$$

The random (Y) in equation (20) follows a skewed heavy-tail distribution (M-VNIG).

7. Kernel Functions and Method of Selecting the Smoothing Parameter:

The kernel functions (nucleus) are used to estimate both regression functions spectral and probability density functions. The method of selecting the introductory parameter (H) is an essential part of estimating the non-parametric regression curve, and choosing the bandwidth parameter is more critical than selecting the kernel function (nucleus). This parameter has several labels, including (constraint capacity - bandwidth - concentration parameter - contrast parameter) and its properties are a non-random, symmetrical, and positive boundary parameter [8]. Table 1 shows the kernel functions used in the search as well as the selection of the introductory parameter based on the thumb rule method [7] [8]:

Table 1: Shows Some Kernel Functions and Methods of Selecting the Smoothing Parameter.

Kernel	K(x)		$h = \hat{\sigma}\, CV(k)\, n^{-\frac{1}{5}}$
Quartic	$(15/16)(1-x^2)^2$	$I(\lvert x\rvert \le 1)$	$CV(k) = 2.78$
Gauss	$(2\pi)^{-0.5}\exp(-x^2/2)$	$I(\lvert x\rvert < \infty)$	$CV(k) = 1.06$

8. Classical Estimation of Multivariate (Non-NR) Functions:

Suppose (n) observations of response variables, non-parametric explanatory variables and the kernel function are available. In that case, the probability function of (Y) that is unconditional on the variable (τ) is challenging to find the most significant potential estimators, So we resort to the concept of mixed distributions in finding these estimators, and, therefore, the probability function of (Y) conditional on (τ) is written as follows:

$$f(Y|\beta,\Sigma,\tau)=\frac{\left(2\left(\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt\right)^2\right)^{-\frac{nR}{2}}\left(\frac{1}{\tau}\right)^{\frac{nR}{2}}}{(\det(\Sigma))^{\frac{n}{2}}\ (\det(\varphi))^{\frac{-R}{2}}\ e^{\frac{1}{2\tau}trace\,(Y-T\beta-\omega\tau)^T\varphi(Y-T\beta-\omega\tau)\Sigma^{-1}}} \qquad (21)$$

Taking the natural logarithm of both sides of equation (21) above and taking the first partial derivative relative to the matrix (β) we get:

$$\frac{\partial \ln f(Y|\beta,\Sigma,\tau)}{\partial\,\beta^T}=-\frac{1}{2\,\tau}trace(-2\,T^T\varphi\,\mathrm{Y}+2T^T\varphi\,\mathrm{T}\,\beta+2\,T^T\varphi\,\omega\,\tau)\Sigma^{-1} \qquad (22)$$

After equalizing equation (22) with zero and performing the integration process relative to the single random variable (τ), we get the most significant potential estimator for (β) as follows:

$$\hat{\beta}_k=\left(\mathrm{T^T}\varphi\,\mathrm{T}\right)^{-1}\mathrm{T^T}\varphi\,\mathrm{Y}-\left(\mathrm{T^T}\varphi\,\mathrm{T}\right)^{-1}\mathrm{T^T}\varphi\,\omega*\frac{\left(\mathrm{H}_\gamma(\mathrm{sqrt}(\alpha\,\vartheta))\right)^{-1}\ (\alpha)^{-0.5}}{(\vartheta)^{-\,0.5}\left(\ \mathrm{H}_{\frac{2(1+\gamma)}{2}}(\mathrm{sqrt}(\alpha\,\vartheta))\right)^{-1}} \qquad (23)$$

To obtain the multivariate (NW-S) and the multivariate (LL-S), we use the relationship: [6]

$$\hat{m}(t,u,h)=e^T\hat{\beta}_k \qquad ,\ e^T=(1,0,0,\dots,0) \qquad (24)$$

If the order of the polynomial (u=0), we have a polyvariate (NW-S) as follows:

$$\hat{m}(t,0,h)=\frac{1}{\sum_{i=1}^n k_h(t-T_i)}\left[\sum_{i=1}^n k_h(t-T_i)\,y_{i1}\,,\dots,\sum_{i=1}^n k_h(t-T_i)\,y_{iR}\ \right]$$

$$-\frac{\frac{\mathrm{H}_{\frac{2(1+\gamma)}{2}}(\mathrm{sqrt}(\alpha\,\vartheta))}{\mathrm{H}_\gamma(\mathrm{sqrt}(\alpha\,\vartheta))}\left(\frac{\alpha}{\vartheta}\right)^{-\,0.5}}{\sum_{i=1}^n k_h(t-T_i)}\left[\sum_{i=1}^n k_h(t-T_i)\,\omega_{i1}\,,\dots,\sum_{i=1}^n k_h(t-T_i)\,\omega_{iR}\ \right] \qquad (25)$$

If $(\alpha>0=\phi\,,\vartheta=0\,,\gamma>0=\phi\,)$ [5], then the (NW-S) of (R) response variables for the model follows the distribution when the error limit of the model follows the (M-VVG) distribution is:

$$\hat{m}(t,0,h)=\frac{1}{\sum_{i=1}^n k_h(t-T_i)}\left[\sum_{i=1}^n k_h(t-T_i)\,y_{i1}\,,\dots,\sum_{i=1}^n k_h(t-T_i)\,y_{iR}\ \right]$$

$$-\frac{\frac{\mathrm{B}_{-\,(1+\phi)}(0)}{2^{\phi}\mathrm{B}_{-\,\phi}(0)}(\phi)^{-\,0.5}}{\sum_{i=1}^n k_h(t-T_i)}\left[\sum_{i=1}^n k_h(t-T_i)\,\omega_{i1}\,,\dots,\sum_{i=1}^n k_h(t-T_i)\,\omega_{iR}\ \right] \qquad (26)$$

Whereas: [10]

$\mathrm{B}_{-\,(\phi)}(0)=\Gamma(\phi)$

If $(\alpha>\ 0=1\,,\vartheta>0=\tilde{\phi}\,,\gamma=-\frac{1}{2})$ [5], then the (NW-S) of (R) response variables for the model follows the distribution when the error limit of the model follows the (M-VNIG) distribution:

$$\widehat{m}(t,0,h) = \frac{1}{\sum_{i=1}^{n} k_h(t-T_i)} \left[\sum_{i=1}^{n} k_h(t-T_i)\, y_{i1}, \dots, \sum_{i=1}^{n} k_h(t-T_i)\, y_{iR} \right] - \frac{\frac{\mathrm{H}_{\frac{1}{2}}\left(\mathrm{sqrt}(\tilde{\phi})\right)}{\mathrm{H}_{-\frac{1}{2}}\left(\mathrm{sqrt}(\tilde{\phi})\right)} \left(\frac{1}{\tilde{\phi}}\right)^{-0.5}}{\sum_{i=1}^{n} k_h(t-T_i)} \left[\sum_{i=1}^{n} k_h(t-T_i)\, \omega_{i1}, \dots, \sum_{i=1}^{n} k_h(t-T_i)\, \omega_{iR} \right] \quad (27)$$

If the rank of the polynomial (u = 1), we have the polyvariate (LL-S) as follows:

$$\widehat{m}(t,1,h) = [1,0]\, \hat{\beta}_m \quad (28)$$

as:

$$T^T \varphi\, \mathrm{T} = \begin{bmatrix} \sum_{i=1}^{n} k_h(t-T_i) & \sum_{i=1}^{n} k_h(t-T_i)\,(t-T_i) \\ \sum_{i=1}^{n} k_h(t-T_i)\,(t-T_i) & \sum_{i=1}^{n} k_h(t-T_i)\,(t-T_i)^2 \end{bmatrix}$$

And based on: [6]

$$\hat{s}_r(t,h) = n^{-1} \sum_{i=1}^{n} k_h(t-T_i)\,(t-T_i)^r \quad (29)$$

$$(T^T \varphi\, \mathrm{T})^{-1} = \frac{\begin{bmatrix} \hat{s}_2(t,h) & -\hat{s}_1(t,h) \\ -\hat{s}_1(t,h) & \hat{s}_0(t,h) \end{bmatrix}}{n\left[\hat{s}_2(t,h)\,\hat{s}_0(t,h) - \left(\hat{s}_1(t,h)\right)^2\right]}$$

The multivariate (LL-S) is as follows:
If ($\alpha > 0 = \phi, \vartheta = 0, \gamma > 0 = \phi$) [5], then the (LL-S) of (R) response variables for the model follows the distribution when the error limit of the model follows the (M-VVG) distribution is:

$$\widehat{m}(t,1,h) = \frac{n^{-1}}{\left[\hat{s}_2(t,h)\,\hat{s}_0(t,h) - \left(\hat{s}_1(t,h)\right)^2\right]} * \left[\sum_{i=1}^{n} k_h(t-T_i)\, y_{i1}\,(\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h)), \dots, \sum_{i=1}^{n} k_h(t-T_i)\, y_{iR}\,(\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h) \right]$$

$$- \frac{\frac{\mathrm{B}_{-(1+\phi)}(0)}{2^{\phi}\mathrm{B}_{-\phi}(0)} (\phi)^{-0.5}\; n^{-1}}{\left[\hat{s}_2(t,h)\,\hat{s}_0(t,h) - \left(\hat{s}_1(t,h)\right)^2\right]} * \left[\sum_{i=1}^{n} k_h(t-T_i)\, \delta_{i1}\,(\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h)), \dots, \sum_{i=1}^{n} k_h(t-T_i)\, \delta_{ik}\,(\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h) \right] \quad (30)$$

If $(\alpha > 0 = 1, \vartheta > 0 = \tilde{\phi}, \gamma = -\frac{1}{2})$ [1], then the (LL-S) of (R) response variables for the model follows the distribution when the error limit of the model follows the (M-VNIG) distribution:

$$\widehat{m}(t,1,h) = \frac{n^{-1}}{\left[\hat{s}_2(t,h)\,\hat{s}_0(t,h) - \left(\hat{s}_1(t,h)\right)^2\right]} * \left[\sum_{i=1}^{n} k_h(t-T_i)\, y_{i1}\, (\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h)), \ldots, \sum_{i=1}^{n} k_h(t-T_i)\, y_{iR}\, (\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h)\right] - \frac{\frac{\mathrm{H}_{\frac{1}{2}}(\mathrm{sqrt}(\tilde{\phi}))}{\mathrm{H}_{-\frac{1}{2}}(\mathrm{sqrt}(\tilde{\phi}))}\left(\frac{1}{\tilde{\phi}}\right)^{-0.5} n^{-1}}{\left[\hat{s}_2(t,h)\,\hat{s}_0(t,h) - \left(\hat{s}_1(t,h)\right)^2\right]} * \left[\sum_{i=1}^{n} k_h(t-T_i)\, \delta_{i1}\, (\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h)), \ldots, \sum_{i=1}^{n} k_h(t-T_i)\, \delta_{ik}\, (\hat{s}_2(t,h) - (t-T_i)\hat{s}_1(t,h)\right] \quad (31)$$

Referring to the third section and using the same method as the seventh section, the estimator of the (ML-M) for the parameter (θ) is as follows**:**

If Y∼M − VVG distribution, then:

$$\widehat{\theta_{ML-M}} = \left(\mathrm{X^TX}\right)^{-1}\mathrm{X^TY} - \left(\mathrm{X^TX}\right)^{-1}\mathrm{X^T}\,\omega\ * \frac{\mathrm{B}_{-(1+\phi)}(0)}{2^{\phi}\mathrm{B}_{-\phi}(0)}(\phi)^{-0.5} \quad (32)$$

If Y∼M − VNIG distribution, then:

$$\widehat{\theta_{ML-M}} = \left(\mathrm{X^TX}\right)^{-1}\mathrm{X^TY} - \left(\mathrm{X^TX}\right)^{-1}\mathrm{X^T}\,\omega\ * \frac{\mathrm{H}_{\frac{1}{2}}(\mathrm{sqrt}(\tilde{\phi}))}{\mathrm{H}_{-\frac{1}{2}}\left(\mathrm{sqrt}(\tilde{\phi})\right)}\left(\frac{1}{\tilde{\phi}}\right)^{-0.5} \quad (33)$$

9. Applied Side:

In this section, what was reached in Section (7) was applied to real data related to Brent crude oil prices, as the data was collected for the period from (2/11/2020) to (8/12/2020) measured in US dollars. The analysis was based on 21 observations (n=21) and used the MATLAB R2019a program. **https://sa.investing.com**

9.1. Identify the Basic Variables and Initialize the Data:

Brent crude oil is one of the most important types used worldwide, especially in the European and African markets. Brent consists of an oil mix of 15 fields, some located in the United States and others in Norway, which produce about (500,000) barrels per day. Based on the differences between it and other crudes, it is generally sold at a higher price than OPEC oil by about a dollar per barrel and at a lower price than West Texas crude by about a dollar. Therefore, in this research, the effect of the closing price, which is the final price at which securities are traded on a given day, will be studied as an explanatory parametric and non-parametric variable on the highest and lowest oil price as response variables (y_1, y_2) respectively. The following figure shows the highest and lowest price of Brent crude oil .

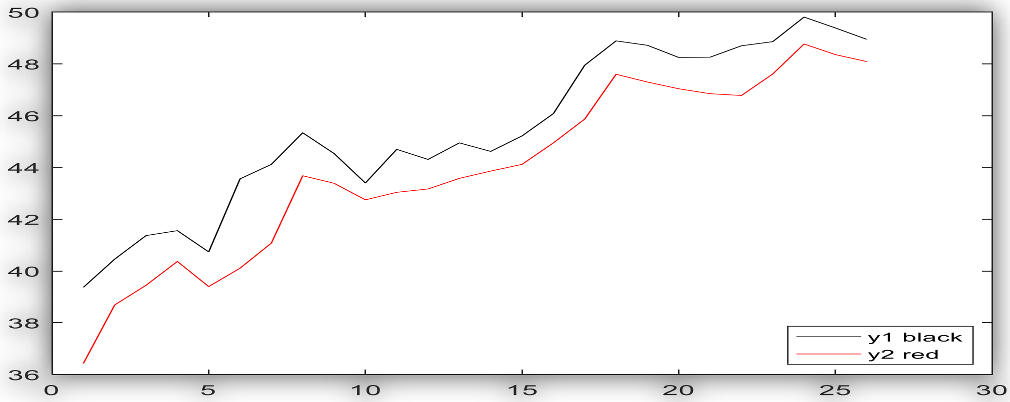

Figure 1: Behaviour of the Highest and Lowest Prices for Brent Crude Oil**.**

Before conducting the data matching process to the study model, represented by the parametric and non-parametric regression model[3]. It was found that the data is twisted towards the left, as the value of the torsion coefficient for the highest price of Brent crude oil (y_1) (-1.2916) and the mutation (1.9611) and the value of the torsion coefficient for the lowest price of Brent crude oil (y_2) (-0.3915) and the mutation (2.1738) and in the case of converting the response variables matrix to a vector operator, i.e., the matrix stacking process, showing the value of the torsion coefficient (-1.9568) and the hyperbolism (3.5478).

To determine the suitability of the oil/Brent crude data to the models used, different sample sizes were used through a good conformity test (Chi-square test) and based on various samples. The table below shows the test values for other samples that led to the matching of the data below the significant level($\alpha = 0.01$).

Table .2. Chi-Square Test Values for Matching Data.

M-VVG Samples $(\alpha, \vartheta, \gamma)$	Chi2-calculate	Chi2-tab. $(\alpha = 0.01)$
(1,0,1)	6.9941	9.2103
(5,0,5)	7.0125	9.2103
(8,0,8)	7.9654	9.2103
(12,0,12)	8.4571	9.2103
(15,0,15)	9.0012	9.2103
M-VNIG Samples $(\alpha, \vartheta, \gamma)$	Chi2-calculate	Chi2-tab. $(\alpha = 0.01)$
$(1,3,-0.5)$	3.5421	9.2103
$(1,6,-0.5)$	4.9567	9.2103
$(1,10,-0.5)$	5.8842	9.2103
$(1,15,-0.5)$	7.6325	9.2103
$(1,20,-0.5)$	8.9874	9.2103

9.2. Estimation of Multivariate (Non-NR) Functions

In this section, the kernel smoothers will be used, represented by the multivariate (NW-S) and the multivariate (LL-S) to estimate the multivariate non-parametric regression function and the (ML-M) to estimate the parametric multivariate regression function parameter based on the samples that led to the matching below the level of significance (α=0.01) and when the model error is distributed distributions of a skewed heavy–tails (M-VVGH) and (M-VNIG), comparison between the smoothers and estimators based on the mean square error criterion. Tables below show the values of the (MSE) for kernel smoothers, and the most significant possibility is estimated.

Table 3. Mean squares error values for kernel smoothers and the maximum likelihood estimator when the model error is distributed skewed heavy-tails (M-VVG) distribution

Regrasion Functions	Multivariate Non-parametric				Multivarite Parametric
	NW-S		LL-S		ML-M
Kernel Function $(\alpha,\vartheta,\gamma)$	Gauss	Quartic	Gauss	Quartic	
(1,0,1)	0.4992	0.5124	0.5235	0.5921	0.6001
(5,0,5)	0.4265	0.4299	0.4421	0.5587	0.5921
(8,0,8)	0.3925	0.4001	0.4100	0.4652	0.4874
(12,0,12)	0.3354	0.3684	0.3847	0.4201	0.4525
(15,0,15)	0.2314	0.2954	0.3125	0.3713	0.4321

Table 4: Values of mean squares error for kernel smoothers and estimator of the maximum likelihood when error models are distributed skewed heavy-tails a heavy-tails (M-VNIG) distribution.

Regression Functions	Multivariate non-parametric				Multivariate parametric
	NW-S		LL-S		ML-M
Kernel Function $(\alpha,\vartheta,\gamma)$	Gauss	Quartic	Gauss	Quartic	
$(1,3,-0.5)$	0.6471	0.6928	0.7120	0.7321	0.7525
$(1,6,-0.5)$	0.6214	0.6425	0.7089	0.7102	0.7432
$(1,10,-0.5)$	0.6001	0.6103	0.6587	0.6623	0.6985
$(1,15,-0.5)$	0.5948	0.5991	0.6325	0.6111	0.6587
$(1,20,-0.5)$	0.5601	0.5812	0.6140	0.6033	0.6433

Table 5: Values of mean squares error for kernel smoothers and the maximum likelihood estimator when model error is distributed Matrix-Variate T (M-VT) distribution at sample $(\alpha = 0, \vartheta = 1, \gamma = -1)$.

Multivariate Non-parametric Regression Function				Multivariate Parametric Regression Function
NW-S		LL-S		ML-M
Gauss Kernel Function	Quartic Kernel Function	Gauss Kernel Function	Quartic Kernel Function	
0.7742	0.7921	0.8012	0.8142	0.8236

Table 6: Values of mean squares error for kernel smoothers and maximum likelihood estimator when model Error is distributed (M-VN) Distribution at sample $(\alpha = 0\ ,\ \vartheta = 51\ , \gamma = -26)$.

Multivariate Non-parametric Regression Function				Multivariate Parametric Regression Function
NW-S		LL-S		ML-M
Gauss Kernel Function	Quartic Kernel Function	Gauss Kernel Function	Quartic kernel function	
0.9210	0.9547	0.9645	0.9754	0.9942

Through table (3) and (4), we notice that whenever the values of the samples increase, values of the mean squares of the error criterion decrease for both smoothers and the method of the maximum likelihood and when the model error is distributed, the distribution of a skewed heavy- tails (M-VVG) and (M-VNIG), as for table (5)and (6), we notice the superiority of the multivariate (NW-S) and the error of the models, which distributes the distribution of (M-VT),(M-VN). Therefore, through tables (3-7), the best smoother was a multivariate (NW-S), and for the fifth sample $(\alpha = 15, \vartheta = 0, \gamma = 15)$ and when the model error is distributed, the distribution of a skewed heavy-tails (M-VVG) and for Gaussian kernel function is as follows:

$$NW - S = \widehat{m}(t, 0, h) = [0.3542 \quad 0.6139]$$

We note from the multivariate (NW-S) that the effect of the closing price of Brent crude oil was (35%) on the highest cost of Brent crude oil and (61%) on the lowest price of Brent crude oil, while (4%) is due to other influences.

Figure 2 shows the actual and prelude values of Brent crude oil's highest and lowest prices.

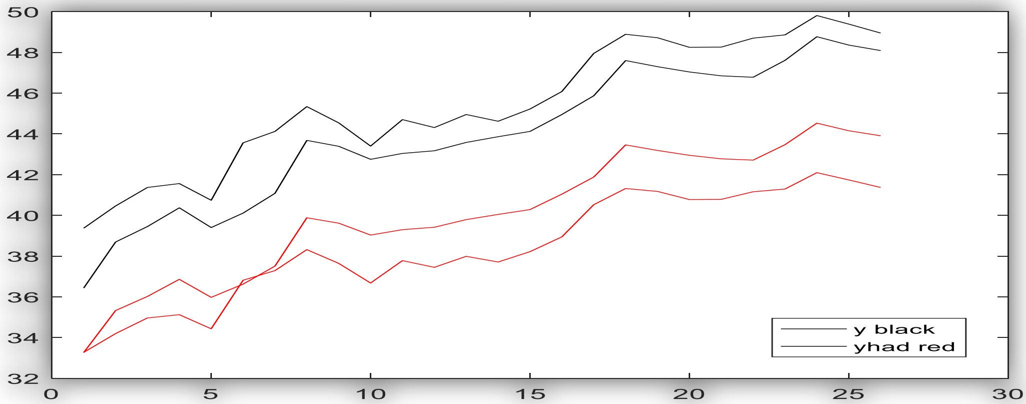

Figure 2: Behaviour of Real and Preliminaries of Brent Crude Oil High and Low Prices.

Conclusions:

The research reached the most important theoretical and applied conclusions, including:

1. When the model error follows the (M-VN) distribution, multivariate (NW-S) and multivariate (LL-S) are similar.
2. The decreasing values of the mean squares error criterion when increasing the values of the additional parameters and for the error of the models skewed heavy-tails (M-VVG) and (M-VNIG) for both smoothers.
3. The multivariate (NW-S) over the multivariate (LL-S) is superior when the error is distributed in the skewed heavy-tails (M-VVG) distribution and for the Gaussian kernel function.
4. The percentage of the impact of the closing price of Brent crude oil on the highest and lowest oil prices is (35% and 61%) respectively, and 4% is due to other external influences.

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